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Some Remarks on Nonlinear Chebyshev Approximation to Functions Defined on Normal Spaces

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1. INTRODUCTION

Let $C_b(X)$ be the space of real-bounded continuous functions defined on a normal space X with the norm

$$\|f\| = \sup\{|f(x)| : x \in X\}$$

and let G be a subset of $C_b(X)$. For $f \in C_b(X)$, $g \in G$, and a real number λ we denote

$$B_\lambda(f, g) = \{x \in X : |f(x) - g(x)| \geqslant \|f - g\| - \lambda\}.$$

DEFINITION 1 (see [7]). G has the weak betweenness property if for any two distinct elements g and h in G and for every nonempty closed subset D of X such that $\inf\{|h(x) - g(x)| : x \in D\} > 0$ there exists a seqeunce $\{g_i\}$ in G such that

(i) $\lim_{i\to\infty} \|g - g_i\| = 0$,

and

(ii) $\inf\{[h(x) - g_i(x)][g_i(x) - g(x)] : x \in D\} > 0$

for all integers i.

DEFINITION 2. An element $g \in G$ is a best approximation to the given $f \in C_b(X)$ when $\|f - g\| \leqslant \|f - h\|$ for all $h \in G$.

We have proved in [7] (generalizing [3, Theorem 1]) the following result: Let us assume that G has the weak betweenness property. Thus, the following theorem holds:

THEOREM 1. *An element $g \in G$ is a best approximation in G to a function*

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$f \in C_b(X)$ if and only if there exists no such element $h \in G$ and such positive $\epsilon < \|f - g\|$ that

$$\inf\{[f(x) - g(x)][h(x) - g(x)] : x \in B_\lambda(f, g)\} > 0$$

for all λ, $0 < \lambda \leqslant \epsilon$.

Remark. We note that Theorem 1 has been formulated in [7] with the following assumption : X is a metric space. However, reviewing [7, proof of Theorem 1] we see that the above assumption can be changed to : X is a topological space.

The main purpose of this paper is to prove that if Theorem 1 holds for every $f \in C_b(X)$ then G must have the weak betweenness property. In the case when X is a compact metric space this fact was established in [6]. An immediate consequence of this fact is that every set G having the betweenness property [1] or being asymptotically convex [5] also has the weak betweenness property.

2. Main Results

THEOREM 2. *If Theorem* 1 *holds for every $f \in C_b(X)$, then G has a weak betweenness property.*

Proof. Let us assume that Theorem 1 holds for every $f \in C_b(X)$ and for a $G \subset C_b(X)$. Let δ_i, $i = 1, 2,\dots$, be a strictly decreasing sequence of positive numbers convergent to zero. Let h, g be two distinct elements in G and let D be a closed subset of X such that

$$\tau = \inf\{|h(x) - g(x)| : x \in D\} > 0.$$

To prove the theorem, we construct the sequence $g_i \in G$, $i = 1, 2,\dots$ such that

(a) $\|g - g_i\| < \delta_i$

and

(b) $\inf\{[h(x) - g_i(x)][g_i(x) - g(x)] : x \in D\} > 0$ for all $i = 1, 2,\dots$.

First, we do this for $i = 1$.

Let

$$Z_1 = \{x \in X : |h(x) - g(x)| \leqslant \tau/2\}$$
$$U_1 = \{x \in X : h(x) - g(x) < -\tau/2\}$$

and $V_1 = X \backslash (Z_1 \cup U_1)$. Obviously D and Z_1 are disjoint closed sets. For all dyadic rationals of the form

$$r = k/2^n, \qquad n = 0, 1,\dots \text{ and } k = 0, 1,\dots, 2^n$$

we define open sets A_r such that

$$Z_1 \subset A_0, \qquad X \backslash D = A_1, \qquad \text{and } \bar{A}_r \subset A_s \qquad \text{for all } r < s.$$

The existence of these sets follows from the normality of the space X and may be proven by induction on n as in [4, pp. 126–127].

Define the nonnegative function p_1 on X such that

$$p_1(x) = 0 \qquad \text{for all } x \in Z_1,$$
$$p_1(x) = \sup\{r : x \notin A_r\}.$$

Now we prove that the function

$$s_1(x) = p_1(x) \operatorname{sign}[h(x) - g(x)] \tag{1}$$

is continuous on X. Let $\epsilon > 0$ and $x \in X$ be arbitrary and let an integer n and a dyadic rational r be such that

$$2^{-n} \leqslant \epsilon \qquad \text{and} \qquad p_1(x) < r < p_1(x) + 2^{-n-1}.$$

Let us define the open set H_x containing x as follows:

$$\begin{aligned} H_x &= (A_r \backslash \bar{A}_{r-2^{-n}}) \cap U_1 && \text{if } x \in U_1, \\ &= (A_r \backslash \bar{A}_{r-2^{-n}}) \cap V_1 && \text{if } x \in V_1, \\ &= A_{2^{-n-1}} && \text{if } x \in Z_1 \end{aligned}$$

where we understood that $A_s = \varnothing$ if $s < 0$ and $A_s = X$ if $s > 1$. Then we have for all $y \in H_x$

$$| s_1(x) - s_1(y)| = | p_1(x) - p_1(y)| < 2^{-n} \qquad \text{if } x \in U_1 \cup V_1$$

and

$$| s_1(x) - s_1(y)| = | s_1(y)| \leqslant | p_1(y)| < 2^{-n} \qquad \text{if } x \in Z_1.$$

Hence the function s_1 is continuous on X.

Define the continuous and bounded function f_1 on X by

$$f_1(x) = g(x) + \mu_1 s_1(x) \tag{2}$$

where

$$0 < \mu_1 < 0.5 \min\{\delta_1, \tau\}. \tag{3}$$

Note that we have

$$\| f_1 - g \| = | f_1(x) - g(x) | = \mu_1 \qquad \text{for all } x \in D. \tag{4}$$

Now we prove that g is not a best approximation to f_1 in G. Because for all $0 < \lambda < \mu_1$ we have

$$B_\lambda(f_1, g) = \{x \in X : \mu_1 \mid s_1(x) \mid \geqslant \mu_1 - \lambda\} = \{x \in X : \mu_1 p_1(x) \geqslant \mu_1 - \lambda\}$$

and hence $B_\lambda(f_1, g) \subset X \backslash A_r$ for all dyadic rationals such that $0 < r < 1 - (\lambda/\mu_1)$ then, for all $x \in B_\lambda(f_1, g)$

$$\begin{aligned}[f_1(x) - g(x)][h(x) - g(x)] &= \mu_1 p_1(x) | h(x) - g(x) | \\ &\geqslant (\mu_1 - \lambda) | h(x) - g(x) | \geqslant (\mu_1 - \lambda)(\tau/2) > 0.\end{aligned}$$

Hence and from Theorem 1 the function g is not a best approximation in G to f_1, i.e., there exists a function $g_1 \in G$ such that

$$\| f_1 - g_1 \| < \| f_1 - g \| = \mu_1 . \tag{5}$$

Hence, from the triangle inequality for a norm and from (3) we have

$$\| g - g_1 \| < \delta_1 , \tag{6}$$

i.e., condition (a) is satisfied for $i = 1$.

Because, from (4) and (5)

$$| f_1(x) - g_1(x) | < | f_1(x) - g(x) | \qquad \text{for } x \in D \tag{7}$$

then for every such x we have

$$\operatorname{sign}[g_1(x) - g(x)] = \operatorname{sign}[f_1(x) - g(x)] = \operatorname{sign}[h(x) - g(x)]. \tag{8}$$

Hence and from (3), (4), and (5) we obtain for all $x \in D$

$$\begin{aligned}&[g_1(x) - g(x)][h(x) - g_1(x)] \\ &\quad = | g_1(x) - g(x) | [h(x) - g_1(x)] \operatorname{sign}[h(x) - g(x)] \\ &\quad = | f_1(x) - g(x) - [f_1(x) - g_1(x)] | (| h(x) - g(x) | - | g_1(x) - g(x) |) \\ &\quad \geqslant (| f_1(x) - g(x) | - | f_1(x) - g_1(x) |)(2\mu_1 - \| f_1 - g_1 \| - \| f_1 - g \|) \\ &\quad = (\mu_1 - | f_1(x) - g_1(x) |)(\mu_1 - \| f_1 - g_1 \|) \geqslant (\mu_1 - \| f_1 - g_1 \|)^2 > 0.\end{aligned}$$

This implies that condition (b) is satisfied and the proof for $i = 1$ is completed.

Because

$$| g_1(x) - g(x) | \geqslant \mu_1 - \| f_1 - g_1 \| > 0$$

for all $x \in D$ then we have

$$\tau_1 = \inf\{| g_1(x) - g(x)| : x \in D\} > 0. \tag{9}$$

Now, replacing h by g_1, τ by τ_1, and δ_1 by δ_2 and using (9) we may construct with the small modifications of above statements the element g_2 in G such that conditions (a) and (b) are satisfied. We do this briefly.

Define the sets

$$\begin{aligned} Z_2 &= \{x \in X : | g_1(x) - g(x)| \leqslant \tau_1/2\}, \\ U_2 &= \{x \in X : g_1(x) - g(x) < -\tau_1/2\}, \\ V_2 &= X \backslash (Z_2 \cup U_2). \end{aligned}$$

Additionally, construct as above the continuous nonnegative function p_2 and set

$$s_2(x) = p_2(x) \operatorname{sign}[g_1(x) - g(x)], \tag{1'}$$

$$f_2(x) = g(x) + \mu_2 s_2(x). \tag{2'}$$

where

$$0 < \mu_2 < 0.5 \min\{\mu_1, \delta_2, \tau_1\}, \tag{3'}$$

$$\| f_2 - g \| = | f_2(x) - g(x)| = \mu_2 \qquad \text{for all } x \in D. \tag{4'}$$

Similarly as above we may prove that there exists $g_2 \in G$ such that

$$\| f_2 - g_2 \| < \| f_2 - g \| = \mu_2 \tag{5'}$$

and

$$\| g - g_2 \| < \delta_2, \tag{6'}$$

i.e., condition (a) is satisfied for $i = 2$.

Because by (4') and (5') it is

$$| f_2(x) - g_2(x)| < | f_2(x) - g(x)| \tag{7'}$$

for all $x \in D$ then we have from (1'), (2'), and (8) that

$$\operatorname{sign}[g_2(x) - g(x)] = \operatorname{sign}[f_2(x) - g(x)] = \operatorname{sign}[h(x) - g(x)]. \tag{8'}$$

Hence and from (3'), (4'), and (5') we obtain

$$\begin{aligned} &[g_2(x) - g(x)][h(x) - g_2(x)] \\ &\quad = | g_2(x) - g(x)|(| h(x) - g(x)| - | g_2(x) - g(x)|) \\ &\quad \geqslant (| f_2(x) - g(x)| - | f_2(x) - g_2(x)|)(| h(x) - g(x)| - | g_2(x) - g(x)|) \\ &\quad \geqslant (\mu_2 - \| f_2 - g_2 \|)(2\mu_1 - \| f_2 - g_2 \| - \| f_2 - g \|) \\ &\quad \geqslant (\mu_2 - \| f_2 - g_2 \|)(2\mu_2 - \| f_2 - g_2 \| - \| f_2 - g \|) \\ &\quad = (\mu_2 - \| f_2 - g_2 \|)^2 > 0. \end{aligned}$$

This implies that condition (b) is satisfied and the proof is completed for $i = 2$.

Because

$$| g_2(x) - g(x)| \geqslant \mu_2 - \| f_2 - g_2 \| > 0$$

for all $x \in D$ then we have

$$\tau_2 = \inf\{| g_2(x) - g(x)| : x \in D\} > 0. \qquad (9')$$

In generally, replacing g_{i-2} by g_{i-1}, τ_{i-2} by τ_{i-1}, and δ_{i-1} by δ_i we may analogously as for $i = 2$ construct $g_i \in G$ for $i = 3, 4,...$ satisfied conditions (a) and (b). Therefore, the proof is completed ▮

COROLLARY 1. *A necessary and sufficient condition that Theorem* 1 *hold for every* $f \in C_b(X)$ *is that G has a weak betweenness property.*

Now we shall give an example of a subset in $C[-1, 1]$ which does not have a weak betweenness property.

EXAMPLE. Let P_2 be the set of all polynomials of degree $\leqslant 2$ and H be the set of so-called H-polynomials [2], i.e., polynomials of the form $\pm (ax^2 + bx + c)^2 + d$ defined on interval $[-1, 1]$. Define $G = P_2 \cup H$. It is known [2] that G is a closed set and that for each function $f \in C[-1, 1]$ there exists the best approximation in G.

We claim that G does not have the weak betweenness property. Let

$$g(x) = (64/45)(x^2 - \tfrac{5}{8})^2 - \tfrac{1}{5},$$
$$h(x) = x.$$

Then

$$g(\tfrac{1}{2}) = g(-\tfrac{1}{2}) = g(1) = g(-1) = 0$$
$$x = 0 \quad \text{and} \quad x = \pm\tfrac{1}{2}(\tfrac{5}{2})^{1/2} - \text{extremal points}$$
$$g(0) = \tfrac{16}{45}, \quad g(\pm\tfrac{1}{2}(\tfrac{5}{2})^{1/2}) = -\tfrac{1}{5}$$
$$g(\tfrac{1}{4}) = h(\tfrac{1}{4}) = \tfrac{1}{4}.$$

Let us set, for example, $D = [-1, 1]\backslash(\frac{1}{5}, \frac{1}{3})$.

Now we prove that there does not exist a sequence of functions $\{g_i\}$ in G lying strictly between $g(x)$ and $h(x)$ for all $x \in D$ and uniformly convergent on D to g. Indeed, such polynomials for sufficiently large i must have four zeroes $x_1 < -1$, $-\frac{1}{2} < x_2 < 0$, $\frac{1}{2} < x_3 < x_4 < 1$ and three extremal points $y_1 \in (x_1, x_2)$, $y_2 \in (x_2, x_3)$, and $y_3 \in (x_3, x_4)$ such that $g_i(y_1) < -\frac{1}{5}$, $g_i(y_2) > 0$ and $-\frac{1}{5} < g_i(y_3) < 0$ (see Fig. 1). This is obviously impossible in P_2.

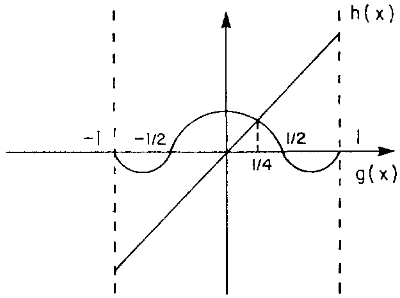

FIGURE 1

Because every H-polynomial in H with three distinct extremal points has such a property that two from these points are zeores of $ax^2 + bx + c$, i.e., two minimum values are equal then such sequence $\{g_i\}$ does not exist also in H. Therefore, G does not have the weak betweenness property. Hence and from Corollary 1 there exist the functions in $C[-1, 1]\backslash G$ for which Theorem 1 does not hold (see also [2]).

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